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Classical and quantum D-branes in 2D string theory

John McGreevy and Herman Verlinde

Department of Physics, Princeton University
Princeton, NJ 08544, U.S.A.
E-mail: mcgreevy@Princeton.edu, verlinde@feynman.princeton.edu

Joerg Teschner

Institut für theoretische Physik, Freie Universität
Berlin, 14195 Berlin, Germany
E-mail: teschner@physik.fu-berlin.de

Abstract: We investigate two classes of D-branes in 2-d string theory, corresponding to extended and localized branes, respectively. We compute the string emission during tachyon condensation and reinterpret the results within the $c = 1$ matrix model. As in [2], we find that the extended branes describe classical eigenvalue trajectories, while, as found in [3], the localized branes correspond to the quantum field that creates and destroys eigenvalues. The matrix model relation between the classical probe and the local collective field precisely matches with the descent relation between the boundary states of D-strings and D-particles.

Keywords: Matrix Models, Tachyon Condensation, D-branes.

Contents

1. Introduction

D-branes are fascinating objects. With the benefit of hindsight, it appears surprising that it took such a long time before the full extent of their role in string theory was appreciated. Undoubtedly, part of their true meaning was obscured by the fact that D-branes start their perturbative life as unnaturally rigid looking objects, with fixed positions or following a fixed classical trajectory. They fully come to life, however, after one realizes that the open/closed string dynamics on their world-volume promotes all their properties, and in particular their positions, into dynamical, quantum mechanical degrees of freedom.

A quantum D-brane not only affects but also reacts to its environment. Including such recoil effects is particularly important in the study of time-dependent phenomena, such as the decay of unstable D-branes into closed strings [1].

A useful arena for studying D-brane dynamics in a fully quantum mechanical context is 2-dimensional bosonic string theory, which is believed to have an exactly soluble dual description in terms of matrix quantum mechanics. In [2] it was proposed that the hermitian $c = 1$ matrix in fact represents the open string tachyon mode of a dense gas of unstable D-particles. In light of this conjecture, it is natural to suspect that the properties and interactions of D-branes in 2-d string theory must have a simple and natural representation in terms of the matrix eigenvalues. With this dictionary in hand, one can then proceed to learn new lessons about the quantum properties of D-branes, some of which hopefully generalize to arbitrary string backgrounds.

In this paper we will investigate two classes of D-branes in 2-d string theory, corresponding to extended and localized branes, respectively. We describe their open string spectrum and the closed string one-point functions on the disk that quantify the perturbative string emission during tachyon condensation. We then reinterpret the results within the $c = 1$ matrix model. We clarify and extend of the result of [2] that the extended branes describe the response to classical probe-eigenvalue trajectories. The localized brane, on the other hand, has the matrix model interpretation as the quantum field that creates and destroys eigenvalues. This interesting result, which was recently obtained in [3], suggests that the localized brane is closely related to the operator that creates the quantum version of the extended brane.

1.1 Classical versus quantum D-branes

For our later discussion, it will be useful to clarify what we mean by classical and quantum D-branes. Abstractly, a D-brane is specified by a conformally invariant boundary state $|B_X\rangle$, which acts as a linear source for all closed string fields ϕ_i:

$$S_{\rm int}(\phi, X) = \sum_i \phi_i \, \langle V_i | B_X \rangle \, . \tag{1.1}$$

Here X denotes some set of parameters that specify the boundary CFT.

This characterization of a D-brane by a boundary state requires that the profiles of the open-string modes on the brane satisfy their classical equations of motion: the D-brane is still classical. Clearly, however, any object responds to its environment, and consistency of string perturbation theory dictates that all brane degrees of freedom must be quantized [4]. The quantization of the collective coordinates of D-branes was further elaborated in [5, 6, 7, 8], and, of course, is crucial for the existence of quantum field theories on the worldvolumes of branes.

For definiteness, let us specialize to the case where the D-brane is a D-particle, and X denotes its position. Its possible boundary states correspond to all possible classical trajectories $X(t)$. To quantize the particle, however, it would be helpful if we could consider arbitrary time dependent trajectories $X(t)$, so that we can write a path-integral expression

JHEP01(2004)039

for its wavefunction as

$$\Psi(X) = \int^X DX(t)\, e^{\frac{i}{\hbar}\left(S(X)+S_{\rm int}(\phi,X)\right)}\,, \tag{1.2}$$

where $S(X)$ is the D-particle worldline action in the string background $\phi_i = 0$. The interaction term $S_{\rm int}$ must be a generalization of (1.1) to general off-shell trajectories, so that we can write

$$S_{\rm int} = \int dt\, L_{\rm int}(\phi, X(t))\,. \tag{1.3}$$

In this paper, motivated by the matrix model formulation of 2-d string theory, we will adopt the following general prescription for obtaining $L_{\rm int}$, which works for general static target space-times. Let $|B_{X^i}\rangle$ denote the boundary state in the space-like CFT describing the D-particle at location X^i, and let $|B^D_{X^0=t}\rangle$ denote the time-like Dirichlet boundary state at $X^0 = t$. Then we define

$$L_{\rm int}(\phi, X(t)) \;=\; \sum_i \phi_i \langle V_i | B_{X(t)}\rangle \tag{1.4}$$

$$|B_{X(t)}\rangle \;=\; |B_{X^i(t)}\rangle \otimes |B^D_{X^0=t}\rangle\,. \tag{1.5}$$

Here in the first factor on the right-hand side, $X_i(t)$ is just a fixed position and not the full time-dependent trajectory; the state $|B_{X^i(t)}\rangle$ defines a consistent boundary CFT. So, via this proposal we can consider arbitrary D-brane trajectories $X(t)$, while keeping the power of exact worldsheet conformal invariance.

The D-particle action and wave-function (1.2) both explicitly depend on the closed string background parameterized by the ϕ_i. Now suppose we promote the ϕ_i to quantum fields, and in addition introduce a multi-particle Hilbert space for the D-particles. In this way, we can try to promote the D-particle wave-function $\Psi(X)$ to a second quantized field operator, which we would like to identify with the "gravitationally dressed" version of the local quantum field that creates and destroys D-particles. This we would like to call the quantum D-brane.

1.2 Organization

The paper is organized as follows. In the next section, we study the Liouville CFT and its boundary states in the limit ($c \to 25$) in which it participates in two-dimensional bosonic string theory. In section 3, we embed this discussion in 2D string theory by studying tensor products of Liouville boundary states with various possible boundary states of the X^0 CFT. In this context, we apply the procedure described in section 1.1 to study D-branes on arbitrary trajectories. Along the way, we perform a (logically independent) Cardy analysis of the spectral density of open-strings associated with the bouncing boundary state, which corroborates our earlier discussion. In section 4 and section 5 we turn to the matrix model description of these processes, using classical and quantum descriptions of probe eigenvalues. We conclude in section 6.

While we were typing this sentence, [9] appeared, which in addition to many other interesting results, identifies in detail a matrix model counterpart for the ZZ states with $m, n > 1$.

JHEP01(2004)039

2. Boundary Liouville CFT

As usual we will define the 2d string theory as the tensor product of the CFT of a free boson with the $c = 25$ Liouville theory. In order to define the latter we shall take the limit $c \downarrow 25$ of the $c > 25$ Liouville theory constructed in [10, 11].

In the semiclassical limit $c \to \infty$ one may describe Liouville theory in terms of the action

$$S_{\rm cl} = \int d^2z \left(\frac{1}{\pi} |\partial_z \varphi|^2 + \mu e^{2b\varphi} \right) . \tag{2.1}$$

The basic fields of the theory are the primary fields

$$V_\alpha(z, \bar{z}) \simeq e^{2\alpha\varphi(z,\bar{z})}, \qquad b \to \infty , \tag{2.2}$$

which have conformal dimensions $\Delta_\alpha = \alpha(Q - \alpha)$. It will be important for us to remember that these fields satisfy the reflection property [12]

$$V_{\frac{Q}{2}+iP}(z, \bar{z}) = R(P) V_{\frac{Q}{2}-iP}(z, \bar{z}) , \tag{2.3}$$

where

$$R(P) = -(\pi\mu\gamma(b^2))^{-\frac{2iP}{b}} \frac{\Gamma(1 + 2ibP)\Gamma(1 + 2ib^{-1}P)}{\Gamma(1 - 2ibP)\Gamma(1 - 2ib^{-1}P)} , \tag{2.4}$$

with

$$\gamma(b^2) \equiv \frac{\Gamma(b^2)}{\Gamma(1 - b^2)} . \tag{2.5}$$

There are subtle quantum modifications of the action (2.1) for finite values of c. In order to describe the behavior of Liouville theory in the weak-coupling asymptotics $\varphi \to -\infty$ one may use the following action (as explained in Part II of [10])

$$S_{\rm q} = \int d^2z \left(\frac{1}{\pi} |\partial_z \varphi|^2 + \mu e^{2b\varphi} + \tilde{\mu} e^{2b^{-1}\varphi} \right) , \tag{2.6}$$

where the coupling constant b is related to c via $c = 1 + 6Q^2$, $Q = b + b^{-1}$, and $\tilde{\mu}(\mu, b)$ is given by

$$\pi\gamma(b^{-2})\tilde{\mu} = (\pi\gamma(b^2)\mu)^{b^{-2}} . \tag{2.7}$$

In order to get a useful perspective on the quantum corrections appearing in (2.6) let us note that

$$S_{\rm q} = \int d^2z \left(\frac{1}{\pi} |\partial_z \varphi|^2 + \mu V_b(z, \bar{z}) \right) . \tag{2.8}$$

The correspondence with (2.6) follows from the observations that for $\varphi \to -\infty$

$$V_\alpha \simeq e^{2\alpha\varphi} + R_\alpha e^{2(Q-\alpha)\varphi} , \qquad R_\alpha \equiv R\left(\frac{i}{2}(Q - 2\alpha) \right) . \tag{2.9}$$

and that $R_b \equiv \tilde{\mu}/\mu$.

We are interested in the limit $c \to 25$, corresponding to $b \to 1$. In order to get finite results for the basic quantities like the reflection amplitude (2.4) or the three-point function [13, 12] we need to keep the combination

$$\mu_{\text{ren}} \equiv \mu\gamma(b^2) \tag{2.10}$$

finite in the limit $b \to 1$. Concerning the action S_{q} let us note that

$$R_b \sim -(\pi\mu_{\text{ren}})^{\epsilon} + \mathcal{O}(\epsilon^2)\,, \qquad \epsilon \equiv 1 - b^2\,, \tag{2.11}$$

which implies

$$\begin{aligned}\mu\left(e^{2b\varphi} + R(b)e^{2b^{-1}\varphi}\right) &\sim \mu\left(e^{-\epsilon\varphi} - e^{\epsilon(\varphi+\ln\pi\mu_{\text{ren}})}\right)e^{2\varphi}\\ &\sim -\mu_{\text{ren}}(2\varphi + \log\pi\mu_{\text{ren}})e^{2\varphi}\,.\end{aligned} \tag{2.12}$$

In this way we have derived some old conjectures concerning the form of the Liouville-interaction for $c = 25$ from the exact solution.

It should be emphasized, however, that (2.12) serves to describe the $c = 25$ Liouville theory *only* in the asymptotics $\varphi \to \infty$; it can not be expected to be exact [10].

2.1 Boundary Liouville theory at $c > 25$

A first important class of conformally invariant boundary conditions for the $c > 25$ Liouville theory may be defined in the semiclassical limit $c \to \infty$ by the boundary action

$$S_{\text{bound}} = \int_{\partial\Gamma} \frac{d\tau}{2\pi} g^{\frac{1}{4}}\left(QK + 2\pi\mu_{\text{B}}e^{b\varphi}\right), \tag{2.13}$$

where τ is a parameter for the boundary, and K is its extrinsic curvature in the background metric g. The parameter μ_{B} which labels the boundary conditions is called the boundary cosmological constant.

The corresponding boundary states were constructed in [14]. They can be represented as

$$|B_s\rangle = \int_{-\infty}^{\infty} \frac{dP}{2\pi}\, e^{-2\pi i P s} v(P)|P\rangle\!\rangle\,, \tag{2.14}$$

where the following definitions have been used: $|P\rangle\!\rangle$ is the Ishibashi-state built from the Virasoro representation with highest weight $\Delta_P = \frac{1}{4}(Q^2 + 4P^2)$, the function $v(P)$ is given as

$$v(P) = i(\pi\mu\gamma(b^2))^{iP/b}\frac{\Gamma(1-2ibP)\Gamma(1-2iP/b)}{P}\,, \tag{2.15}$$

and the parameter s used in (2.14) is related to the boundary cosmological constant μ_{B} via

$$\cosh\pi bs = \frac{\mu_{\text{B}}}{\sqrt{\mu}}\sqrt{\sin\pi b^2}\,. \tag{2.16}$$

Using the parameter s instead of μ_{B} turns out to be rather natural for the description of boundary Liouville theory.

In order to construct open string theories involving a Liouville direction, one needs to consider Liouville theory on the strip $[0, \pi]$ with boundary conditions labelled by parameters

JHEP01(2004)039

s_2 and s_1. The spectrum $\mathcal{H}^{\mathrm{B}}_{s_2 s_1}$ of Liouville theory on the strip can be deduced from the bulk spectrum via world-sheet duality [15]. It always contains a continuous part given by $\int_0^\infty dP\ \mathcal{V}_{P,c}$, where $\mathcal{V}_{P,c}$ is the highest weight representation of the Virasoro algebra with weight Δ_P and central charge c. In addition to the bulk primary fields $V_\alpha(z,\bar{z})$ one may now also consider the boundary fields $\Psi^{s_2 s_1}_\alpha(\tau)$ which create the states $|\alpha; s_2, s_1\rangle_{\mathrm{B}}$ in $\mathcal{H}^{\mathrm{B}}_{s_2 s_1}$ from the vacuum. The boundary fields are fully characterized by their covariance w.r.t. conformal transformations together with their three point function on the disk [16]. They satisfy a reflection property analogous to (2.3):

$$\Psi^{s_2 s_1}_{\frac{Q}{2}+iP}(\tau) = R(P|s_2,s_1)\Psi^{s_2 s_1}_{\frac{Q}{2}-iP}(\tau)\,, \tag{2.17}$$

where the boundary reflection amplitude $R(P|s_2,s_1)$ is given by the expression [14]

$$\begin{aligned} R\left(P|s_2,s_1\right) = \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{-\frac{iP}{b}}\frac{\Gamma_b(+2iP)}{\Gamma_b(-2iP)}\times \\ \times\frac{S_b(\frac{Q}{2}-i(P+s_1+s_2))S_b(\frac{Q}{2}-i(P+s_1-s_2))}{S_b(\frac{Q}{2}+i(P-s_1-s_2))S_b(\frac{Q}{2}+i(P-s_1+s_1))}\,. \end{aligned} \tag{2.18}$$

Integral representations for the special functions G_b and S_b can be found e.g. in [14]. For our purposes it will be enough to note that $G_b(x)$ and $S_b(x)$ are analytic in the strip $0 < \mathrm{Re}(x) < Q$ and have a simple pole at $x = 0$.

Another very interesting class of boundary states were found in [17]. At present we only have a physical interpretation for the simplest of these boundary states, which will be denoted $|B_{\mathrm{ZZ}}\rangle$. It is given by an expression of the following form:

$$|B_{\mathrm{ZZ}}\rangle = \int_{-\infty}^{\infty}\frac{dP}{2\pi}\,U_{\mathrm{ZZ}}(P)|P\rangle\!\rangle, \tag{2.19}$$

where the one-point function $U_{\mathrm{ZZ}}(P) \equiv \langle P|B_{\mathrm{ZZ}}\rangle$ is

$$U_{\mathrm{ZZ}}(P) = \frac{(\pi\mu\gamma(b^2))^{(2iP-Q)/2b}\Gamma(bQ)\Gamma(Q/b)Q}{2iP\,\Gamma(2ibP)\Gamma(2iP/b)}\,. \tag{2.20}$$

The boundary conditions described by (2.19) have the remarkable property that the spectrum of Liouville theory on the strip with boundary conditions corresponding to $|B_{\mathrm{ZZ}}\rangle$ on both sides contains the Virasoro representation of the identity *only* [17].

2.2 Boundary Liouville theory at $c = 25$

Our aim in this subsection is to discuss the limit $b \to 1$ of boundary Liouville theory. Taking this limit is unproblematic in the case of $|B_{\mathrm{ZZ}}\rangle$. The one-point function for the ZZ state becomes

$$U_{ZZ}(P) = \frac{2}{i\pi}e^{i\delta(2P)}\sinh 2\pi P\,, \tag{2.21}$$

where the phase

$$e^{i\delta(P)} = (\pi\mu_{\mathrm{ren}})^{-iP/2}\frac{\Gamma(1+iP)}{\Gamma(1-iP)} \tag{2.22}$$

is known as the legpole factor.

However, taking $b \to 1$ requires a bit of care in the case of the states $|B_s\rangle$. Let us first of all note that in the limit $b \to 1$ the equation (2.16) becomes

$$\cosh \pi s \simeq \mu_{\rm B} \sqrt{\frac{\epsilon}{\mu_{\rm ren}}} \sqrt{\epsilon} \equiv \frac{\mu_{B,\rm ren}}{\sqrt{\mu_{\rm ren}}} \,, \tag{2.23}$$

introducing a renormalized boundary cosmological constant $\mu_{B,\rm ren}$ that corresponds to finite values of s. A useful perspective on the origin of the renormalization of $\mu_{\rm B}$ can be gained from the following considerations. The semiclassical field $e^{b\varphi(\tau)}$ that appears in the boundary action gets replaced by the primary boundary field $\Psi_b^{s,s}(\tau)$ in the quantum theory. Indeed, taking into account the reflection relation (2.17) we may observe that $\Psi_b^{s,s}(\tau)$ is the *unique* primary boundary field with conformal dimension 1. We may then notice that for $\varphi \to -\infty$

$$\Psi_b^{s,s}(\tau) \simeq e^{b\varphi} + R_{b,s}\; e^{b^{-1}\varphi} \,, \qquad R_{\alpha,s} \equiv R\left(\frac{i}{2}(Q-2\alpha); s, s\right) . \tag{2.24}$$

Upon taking the limit $b \to 1$ we now have

$$R(b|s,s) \sim -(\pi\mu_{\rm ren})^{\epsilon/2} + \mathcal{O}(\epsilon^2) \,. \tag{2.25}$$

The important minus sign in front comes from the factor $\Gamma_b(2iP)/\Gamma_b(-2iP)$ in the boundary reflection amplitude. Inserting this into (2.24) yields

$$\begin{aligned} \mu_{\rm B}(e^{b\varphi} + R(b|s,s)e^{b^{-1}\varphi}) &\sim \mu_{\rm B}\left(e^{-\frac{1}{2}\epsilon\varphi} - e^{\frac{1}{2}\epsilon(\varphi + \log \pi\mu_{\rm ren})}\right) e^{\varphi} \\ &\sim -\mu_{B,\rm ren}\left(\varphi + \frac{1}{2}\log \pi\mu_{\rm ren}\right) e^{\varphi} \,. \end{aligned} \tag{2.26}$$

Remarks.

1. More formally one may deduce the need for renormalization of $\mu_{\rm B}$ from the fact that $\Psi_b^{s,s}$ vanishes as $1 - b^2$ for $b \to 1$, as follows from the results of [16]. We may then define $\mu_{B,\rm ren}$ such that

$$\lim_{b\to 1} \mu_{\rm B} \Psi_b^{s,s}(x) = \mu_{B,\rm ren} \Psi_1'^{\,s,s}(x) \,, \qquad \Psi_1'^{\,s,s}(x) \equiv [\partial_\alpha \Psi_\alpha^{s,s}(x)]_{\alpha=1} \,. \tag{2.27}$$

2. We would like to emphasize that the existence of a continuous spectrum in $\mathcal{H}^{\rm B}_{s_2 s_1}$ is unaffected by the limit $b \to 1$. It follows rather robustly from the pole at $P = 0$ in the boundary state (2.14). The corresponding divergence of the annulus amplitude is related by world-sheet duality to the usual volume divergence that signals the presence of continuous spectrum in the open string channel; see e.g. [18] for a more detailed discussion in a very similar context. The results of [15] furthermore imply absence of bound states in $\mathcal{H}^{\rm B}_{s_2 s_1}$ as long as s_i, $i = 1, 2$, correspond to positive values of $\mu_{B,\rm ren}$, i.e.

$$\mathcal{H}^{\rm B}_{s_2 s_1} = \int_0^\infty dP\, \mathcal{V}_{P,25} \,, \tag{2.28}$$

where $\mathcal{V}_{P,c}$ denotes the Virasoro highest-weight representation with central charge $c = 1 + 6Q^2$ and highest weight $\Delta_P = \frac{1}{4}Q^2 + P^2$.

3. Just as for (2.12), we should emphasize that (2.26) is *not* expected to be accurate at finite values of φ; we only expect it to be useful in the asymptotics $\varphi \to -\infty$.

4. Nevertheless we may infer the main qualitative features of the boundary interaction as follows. If the world-sheet is represented as the upper half-plane we may represent the boundary condition as

$$i(\partial - \bar{\partial})\varphi(x) = 2\pi b \mu_{B,\text{ren}} \Psi_1'^{\,s,s}(x)\,, \tag{2.29}$$

where $\Psi_1'^{\,s,s}(x)$ was defined in (2.27) and $\varphi(z,\bar{z}) \equiv \frac{1}{2}[\partial_\alpha V_\alpha(z,\bar{z})]_{\alpha=0}$. Equation (2.26) implies that the boundary conditions tend to Neumann-type boundary conditions in the limit $\varphi \to -\infty$. On the other hand let us note that the reflection property (2.17) implies that the boundary potential still has the property that it is fully reflecting.[1]

3. D-branes in 2d string theory

3.1 Extended D-branes

A certain class of D-branes in the two-dimensional string theory may be described by boundary potentials of the form

$$T_{\text{roll},\nu}(X^0,\varphi) = e^{\nu X^0} e^{2(1-\nu)\varphi}\,, \qquad T_{\text{bounce},\nu}(X^0,\varphi) = \cosh(\nu X^0) e^{2(1-\nu)\varphi}\,. \tag{3.1}$$

Our aim will be to gain insight into the space-time interpretation of these branes. Of course one can only hope for an exact solutions of the corresponding boundary CFT's for particular choices of the parameter ν, like $\nu = 0$ or $\nu = 1$. Linear combinations of the resulting potentials will correspond to boundary states of the form

$$|B\rangle = |B_{X^0}\rangle \otimes |B_s\rangle\,, \tag{3.2}$$

where $|B_{X^0}\rangle$ is some boundary state in the free boson CFT that corresponds to one of the following boundary potentials

$$T_{\text{static}}(X^0) = 0\,, \qquad T_{\text{bounce}}(X^0) = \lambda \cosh X^0\,, \qquad T_{\text{roll}}(X^0) = \lambda e^{X^0}\,. \tag{3.3}$$

These boundary states of the X^0 CFT have been the subject of active investigation [1].

The space-time interpretation of the corresponding branes is of course simplest in the case of $T_{\text{static}}(X^0)$, where the time direction is represented simply by a free boson with Neumann type boundary conditions. It has spectrum $\int_{-\infty}^{\infty} d\omega \mathcal{V}_{\omega,1}$. Tensoring with Liouville theory yields a continuous spectrum of open strings parametrized by the real half-line.

Our previous discussion of boundary Liouville theory then provides the space-time interpretation of these branes. Let us recall that the boundary conditions that correspond to the $|B_s\rangle$-boundary states tend to the the Neumann boundary conditions for $\varphi \to -\infty$. In this limit we therefore find D-strings stretched along the φ-direction. The reflection

[1] Strictly speaking, we should tune the bulk potential to zero to make this assertion. By using the results of [16] it is possible to show that taking $\mu \to 0$ preserves the reflection relation (2.17).

property (2.3) implies that the force that acts upon the end points of the open strings ultimately pushes all of them into the weak coupling region $\varphi \to -\infty$. It is natural to interpret this fact by saying that the D-strings gradually disappear when we go into the strong coupling region $\varphi \to +\infty$: we need open strings with very high energies in order to probe deeply into $\varphi \to +\infty$.

To generalize slightly one may imagine taking the limit where $\nu \to 0$ in the potentials (3.1). Since the variation with X^0 becomes very slow one would expect that this limit corresponds to the adiabatic approximation. This may be supported by observing that X^0 is related to the corresponding classical field $X^0_{\rm cl}$ by a rescaling $X^0_{\rm cl} \equiv \sqrt{\hbar} X^0$. Identifying $\nu \equiv \sqrt{\hbar}$ we observe that sending $\nu \to 0$ should allow us to replace νX^0 by its classical value $X^0_{\rm cl}$. We are thereby lead to expect that the adiabatic approximation simply means making the boundary cosmological constant $\mu_{B,\rm ren}$ time-dependent. We will return to this situation in section 3.3.

The situation is somewhat more subtle in the cases where the boundary potential involves $T_{\rm bounce}(X^0)$ or $T_{\rm roll}(X^0)$. The work of [19, 20] gives good control over the corresponding euclidean theories, but the analytic continuation of these results to the corresponding timelike CFT's is subtle and does not seem to be fully understood. In section 3.4 we will discuss this analytic continuation for the case of $T_{\rm bounce}(X^0)$. For the time being let us consider the case of $T_{\rm roll}(X^0)$, which seems much simpler in this respect. In this case we have the one-point function [21]

$$\langle \omega | B_\lambda \rangle_{\rm roll} = \pi \frac{\lambda^{-i\omega}}{\sinh(\pi\omega)}\,, \qquad (3.4)$$

which displays a pole at $\omega = 0$. Following our remark 2 at the end of the previous section we therefore expect that the spectrum contains $\int_0^\infty d\omega \mathcal{V}_{\omega,1}$. This would imply the existence of a continuous spectrum of on-shell open strings. We will return to the space-time interpretation of $T_{\rm roll}(X^0)$ in section 3.3.

3.2 Remarks on the tension of D-strings

A reasonable concept of "mass" or "tension" for these branes is not obvious; standard discussions such as [22] are not applicable here. Instead we shall propose the following arguments.

After all, our D-strings are not homogeneous. In the weak coupling region $\varphi \to -\infty$ they are just like ordinary D-strings, but if we go down to strong coupling $\varphi \to \infty$ the D-strings gradually disappear: open strings will not have their end-points in this region, as these end points would feel a strong force pushing them into the weak coupling region.

We should therefore not expect the corresponding density of tension to be homogeneous either. It should become the usual constant tension per length of the D-string for $\varphi \to -\infty$ and should vanish for $\varphi \to \infty$. To have an estimate for the total tension of the D-string we first of all need to regularize the infinity from $\varphi \to -\infty$ by introducing some cut-off Λ. We may then try to estimate the tension by replacing the boundary potential that makes the brane disappear by a reflecting wall at $\varphi = \varphi_m$ with $\varphi_m(\mu_{B,\text{ren}})$ given by

$$\mu_{B,\text{ren}} \left(\varphi_m + \frac{1}{2} \log(\mu_{\text{ren}}) \right) e^{\varphi_m} = \sqrt{\mu_{\text{ren}}} \,. \tag{3.5}$$

This estimate may not be accurate quantitatively, but qualitatively it seems clear that large values of $\mu_{B,\text{ren}}$ imply large negative values of φ_m.

In order to support our proposal let us consider the overlap between the boundary state $|B_s\rangle$ and a closed string wave-packet $\langle\Psi|$ that decays exponentially for $\varphi \to -\infty$:

$$\langle\Psi|B_s\rangle = \int_0^\infty \frac{dP}{2\pi} \bar{\Psi}(P) \langle P|B_s\rangle \,. \tag{3.6}$$

Exponential decay for $\varphi \to -\infty$ implies that $\bar{\Psi}(P)$ is analytic in some strip around the real P-axis. The reflection property (2.3) furthermore implies that

$$\bar{\Psi}(P) = \frac{1}{2}(\bar{\Psi}(P) + R(P)\bar{\Psi}(P)) \sim (\text{const.})P \qquad \text{for } P \to 0 \,, \tag{3.7}$$

so that the pole of $\langle P|B_s\rangle$ gets cancelled. In the limit $s \to \infty$ corresponding to $\mu_{B,\text{ren}} \to \infty$ we therefore get $\langle\Psi|B_s\rangle \to 0$ from the factor $e^{-2\pi i s P}$ in (2.14). This means that the coupling to *all* closed string wave packets that decay fast in the weak coupling region $\varphi \to -\infty$ goes to zero if we raise the boundary cosmological constant to infinity.

We are thereby lead to the proposal that increasing $\mu_{B,\text{ren}}$ (to infinity) decreases the "tension". This would imply that the D-strings are instable against processes which increase $\mu_{B,\text{ren}}$. The rolling tachyon discussed in [2] is such a process as we will discuss more explicitly in the following subsection.

3.3 Rolling tachyons: worldsheet treatment

In order to give an alternative description for the time dependence of the background characterized by $T_{\text{roll}}(X^0) = \lambda e^{X^0}$ let us study a D-string with time-dependent $\mu_{B,\text{ren}} \equiv z(t) = \lambda e^t$, considered as an external source for closed strings. A natural ansatz for the emission amplitude at a *fixed* time t is given by the overlap $\langle\omega|B_t\rangle$, where

$$|B_t\rangle = |B_t^{\text{D}}\rangle_{X^0} \otimes |B_{s(t)}\rangle \,, \qquad \langle\omega| = (\langle\omega|_{X^0} \otimes \langle P|)_{P=\omega} \,. \tag{3.8}$$

In (3.8) we have denoted the boundary state that realizes Dirichlet boundary conditions for the time direction by $\langle B_t^{\text{D}}|$,

$$\langle\omega|B_t^{\text{D}}\rangle_{X^0} = e^{-i\omega t} \,. \tag{3.9}$$

The total amplitude for closed string emission is then given as

$$\mathcal{A}(\omega) = \int_{-\infty}^{\infty} dt \, \langle\omega|B_t\rangle = \int_{-\infty}^{\infty} dt \, e^{i(\delta(\omega) - \omega t)} \frac{\cos \pi\omega \, s(t)}{\sinh \pi\omega} \,, \tag{3.10}$$

where $s(t)$ parameterizes the probe trajectory via

$$z(t) = \sqrt{\mu} \cosh \pi s(t) \,. \tag{3.11}$$

The rest of the calculation proceeds as in [2]: let us change variables to $t = t(s)$, defined by $\lambda e^{t(s)} = \sqrt{\mu}\cosh \pi s$. In doing this we pick up a measure

$$\rho(s) = \frac{dt}{ds} = \frac{\pi}{1+e^{2\pi s}} - \frac{\pi}{1+e^{-2\pi s}}\,, \tag{3.12}$$

so that

$$\mathcal{A}(\omega) = \int_{-\infty}^{\infty} ds\, \rho(s) e^{i(\delta(\omega)-\omega t(s))} \frac{\cos \pi\omega s}{\sinh \pi\omega}\,. \tag{3.13}$$

We now comment on the form of this integral.

The first thing we notice is that, as it stands, (3.13) is infinite: the integrand at late and early times reduces to a constant plus an oscillating piece, and the constant piece leads to a linearly divergent contribution. The physical origin of this divergence is that we are in effect sitting on top of a resonance. We can regulate it by going a little bit off-shell, replacing $e^{-i\omega t}$ by $e^{-i\tilde{\omega} t}$ with $\tilde{\omega}$ slightly different from ω. After this, it should be possible to evaluate the integral by contour deformation. We leave this task for the future.

It is instructive to compare the integral (3.13) with the emission amplitude due to the on-shell boundary CFT description of the same tachyon condensation process. Consider the special boundary state $|B_{s=i/2}\rangle$ tensored with the rolling tachyon state. The corresponding on-shell production amplitude

$$\tilde{\mathcal{A}}(\omega) = \left[\langle \omega | B_\lambda \rangle_{X^0} \langle P | B_{i/2} \rangle\right]_{2P=\omega}\,, \tag{3.14}$$

is equal to

$$\tilde{\mathcal{A}}(\omega) = \frac{\pi e^{i\delta(\omega)}}{\sinh(\pi\omega/2)\sinh(\pi\omega)}\,. \tag{3.15}$$

We notice that this amplitude is the same (up to an irrelevant overall factor $e^{i\omega\infty}$) as the answer one would get by replacing the integral in (3.13) by a sum of the residues at the poles in $\rho(s)$ at $s = s_n \equiv \frac{i}{2}(2n+1)$, cf. [2]. The integrand in (3.13), however, also has branch cuts that (most likely) will spoil a precise correspondence with the on-shell amplitude (3.15). This is to be expected, since the two descriptions of the tachyon condensation process, though very similar, are in fact different.

Our adiabatic description (3.10) of the rolling tachyon has the attractive property that we can "see" the tachyon condensing by building up a growing open string tachyon potential on the D-strings. Comparing with the matrix model calculation in [2] (see also section 4.2) furthermore makes the identification between the boundary cosmological constant $\mu_{B,\mathrm{ren}}$ and the eigenvalue coordinate z manifest.

Taking into account the discussion from the previous subsection we now also see what the final state is: the D-string has disappeared altogether. This strongly suggests that there cannot be propagating on-shell open strings in the final state, although they may have been present in the initial state: there is no brane left to support open strings for $t \to \infty$. We also see what has happened to the on-shell open strings that may have been present in the initial state: they were pushed out to $\varphi \to -\infty$ during the process of tachyon condensation.

JHEP01(2004)039

3.4 Remarks on the tachyon bounce

In this subsection, we will make a few somewhat speculative comments about the open string spectrum on a D-string in the presence of an open string tachyon that follows the bounce trajectory.

To begin with, let us recall [21] that the definition of the one-point function $\langle\omega|B_\lambda\rangle_{\rm bounce}$ is not unique and requires the choice of a contour. Two natural choices of contour were discussed in [21], leading to the results

$$\langle\omega|B_\lambda\rangle^{\rm HH}_{\rm bounce} = \pi\frac{\hat{\lambda}^{-i\omega}}{\sinh(\pi\omega)}\,, \qquad \text{and} \qquad \langle\omega|B_\lambda\rangle^{\rm real}_{\rm bounce} = 2\pi\frac{\sin(\omega\ln\hat{\lambda})}{\sinh(\pi\omega)}\,, \tag{3.16}$$

respectively, with $\hat{\lambda} \equiv \sin\pi\lambda$. It would clearly be of interest to know the spectrum of the X^0-CFT on the strip with boundary conditions characterized by $|B_\lambda\rangle_{\rm bounce}$. We have previously argued that there exists a continuous part in the spectrum if the one-point function $\langle\omega|B_\lambda\rangle_{\rm bounce}$ has a pole at $\omega = 0$. In this respect it is striking to observe that $\langle\omega|B_\lambda\rangle^{\rm HH}_{\rm bounce}$ has such a pole, whereas $\langle\omega|B_\lambda\rangle^{\rm real}_{\rm bounce}$ does not have a pole at $\omega = 0$.

A further hint about the spectrum of the X^0-CFT on the strip is obtained by studying the analytic continuation of the partition function of the corresponding euclidean theory, which was calculated in [23, 19]. The result was

$$Z_{\rm euc}(g,q) = \frac{1}{\eta(q)}\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\sum_{n\in\mathbb{Z}} q^{(n+\beta/4\pi)^2}\,, \tag{3.17}$$

where β is related to ϕ via: $\sin(\beta/4) = \cos(\pi g)\sin(\phi/2)$. We can rewrite (3.17) as

$$Z_{\rm euc}(g,q) = \int_{-\infty}^{\infty} d\sigma\,\rho_{\rm euc}(\sigma)\frac{q^{(\sigma/4\pi)^2}}{\eta(q)} \tag{3.18}$$

where the variable σ is a continuation of β to the real line. The spectral density is

$$\rho_{\rm euc}(\sigma) = \frac{d\phi}{d\sigma} = \mathrm{Re}\,\frac{1}{2}\frac{|\cos\sigma/4|}{\sqrt{\bar{\lambda}^2 - \sin^2\sigma/4}} \tag{3.19}$$

where $\bar{\lambda} \equiv \cos\pi g$. As observed in [23, 19], the density of open-string states has support in an infinite series of finite bands. The bands connect into a single continuous spectrum at $g = 0$, when the boundary CFT represents a Neumann boundary condition, and degenerate to a discrete spectrum for $g = 1/2$.

It seems natural to try and construct the partition function of the corresponding minkowskian theory by contour rotation. In order to do this, one first of all has to represent (3.18) as a linear combination of integrals of (3.19) over contours that pass the branch cuts in the upper or lower half-plane. Naive continuation of (3.19) to $\sigma/4 = i\pi s$ yields

$$\rho(s) = \frac{\pi\cosh\pi s}{\sqrt{\sinh^2\pi s - \bar{\lambda}^2}} \tag{3.20}$$

where $\bar{\lambda} = \cos\pi\lambda$. It is tempting to view $\rho(s)$ as the spectral density of the minkowskian theory. As a warning, however, we should add that one may have to face the possibility that the final outcome for the minkowskian partition function critically depends on how one has chosen the contour. Indeed we expect that the definition of the Minkowski boundary

CFT is not unique, but rather requires additional input. This is natural from the point of view of the minisuperspace analysis of [24], which suggests that the non-uniqueness in the definition of the boundary CFT corresponding to $T_{\rm bounce}(X^0)$ is directly related to the non-uniqueness of the vacuum in general time-dependent backgrounds.[2]

In a later section, we will find an expression very similar to (3.20) from studying a classical rolling eigenvalue in the matrix model. We expect that a comparison between the CFT and matrix model result will provide useful physical guidance in finding a precise definition of the Minkowski theory in the presence of the tachyon bounce.

3.5 Localized D-branes

We will now briefly discuss the interpretation of the boundary states proposed in [17]. In the semiclassical limit, $b \to \infty$, the interpretation of these boundary states is clear: they describe branes localized in the strong coupling region $\varphi \to \infty$. This can be seen from the fact that, in this limit, the one-point function of the closed-string tachyon on the disk with these boundary conditions diverges at the boundary of the disk.

However, the point $b = 1$ is in some sense quite far from the semiclassical limit $b \to 0$. We would therefore like to convince ourselves that the main features of the interpretation above persist for $b = 1$. To this aim let us consider $\langle \Psi_t | B_{\rm ZZ} \rangle$ for a time-dependent[3] wave-packet

$$\langle \Psi_t | B_{\rm ZZ} \rangle = \int_0^\infty \frac{dP}{2\pi}\, e^{-i\Delta_P t}\, \bar{\Psi}(P) \langle P | B_{\rm ZZ} \rangle\,, \tag{3.21}$$

where $\Delta_P = \frac{1}{4}(Q^2 + 4P^2)$. As discussed in some detail in [10], $\langle \Psi_t |$ describes a wave-packet that comes in from $\varphi \to -\infty$ for $t \to -\infty$ and gets reflected back into $\varphi \to -\infty$ for $t \to +\infty$. With the help of the method of stationary phase it is easy to see that the vanishing of the one-point function $\langle P | B_{\rm ZZ} \rangle$ at $P = 0$ implies very rapid decay of $\langle \Psi_t | B_{\rm ZZ} \rangle$ for $t \to \infty$. In other words, the coupling of closed strings to $| B_{\rm ZZ} \rangle$ decays very rapidly for $\varphi \to -\infty$.

On the other hand, we may note that the average energy $\bar{\Delta}_P$ of a wave-packet $\langle \Psi |$ gives a measure for how deeply the packet penetrates into the strong coupling region. The one-point function $\langle P | B_{\rm ZZ} \rangle$ diverges like $e^{\pi P}$ for $P \to \infty$. This means that $\langle \Psi | B_{\rm ZZ} \rangle$ grows exponentially if we increase $\bar{\Delta}_P$. In other words, the deeper we probe into the strong coupling region, the stronger we "feel" the presence of the brane.

These localization properties, together with the fact that the open string spectrum on the branes described with the help of $| B_{\rm ZZ} \rangle$ is (almost) trivial makes them natural candidates for the degrees of freedom that define the holographic dual of the 2d closed string background.

[2]In view of this comment it is natural to ask why we did not encounter such ambiguities in the case of $T_{\rm roll}(X^0)$. Our remark from the end of the previous subsection offers an explanation: as the D-string has disappeared after the tachyon has condensed, we can not have outgoing on-shell open strings for $t \to \infty$. This corresponds to a preferred choice of boundary condition for the Klein-Gordon operator in the mini-superspace analysis of [24], and therefore to a preferred choice of the vacuum. This offers an explanation for why the situation seemed to be much clearer in the case of $T_{\rm roll}(X^0)$.

[3]For the following Gedankenexperiment we are not talking about 2d string theory, we just consider the Liouville CFT.

JHEP01(2004)039

3.5.1 Closed-string emission from the rolling D-particle

Consider the $(1,1)$ ZZ state tensored with the bouncing boundary state in the X^0 CFT, with the Hartle-Hawking prescription. The amplitude for emission of an on-shell closed string from such a brane is [21, 17]

$$\mathcal{A}(\omega, P) = \langle \omega | B_\lambda \rangle^{\mathrm{HH}}_{\mathrm{bounce}} \langle P | B_{ZZ} \rangle = 2i\sqrt{\pi}\, \hat{\lambda}^{-i\omega} e^{i\delta(2P)} \frac{\sinh 2\pi P}{\sinh \pi\omega}\,, \tag{3.22}$$

which upon setting $\omega = 2|P|$ becomes[4]

$$\mathcal{A}(\omega) = \left[\langle \omega | B_\lambda \rangle^{\mathrm{HH}}_{\mathrm{bounce}} \langle P | B_{ZZ} \rangle\right]_{\omega = 2|P|} = 2i\sqrt{\pi}\, e^{i\omega \ln \hat{\lambda}} e^{i\delta(\omega)}\,. \tag{3.23}$$

Other than the leg-pole factor, the wavefunction of this source is a plane-wave in momentum space. As a result, the eigenvalue-space profile (obtained by stripping off the leg-pole factor and fourier transforming) is localized in space and time. Reading off the time-delay from the phase factor, the source is localized at the point in spacetime where the eigenvalue reaches the turning point. This observation implies that we can think of the ZZ state as a local source which initializes the rolling trajectory.[5]

4. Rolling eigenvalues: classical treatment

In the next two sections we will study within the $c = 1$ matrix model, the creation of closed string modes due to an extra rolling matrix eigenvalue. We will consider two situations: a classical probe and a quantum probe. The classical probe is defined in direct analogy with classical D-branes, and corresponds to an extra eigenvalue that follows a prescribed classical trajectory. As we will show in the next section, the quantum probe is obtained from the classical probe by applying standard path-integral quantization. Our goal is to understand the matrix model calculations in terms of the CFT boundary states. We start with the classical situation.

4.1 Adding a classical eigenvalue

Recall that the matrix quantum mechanics can be solved via a path-integral (see e.g. [25]), by first discretizing time and starting from the (semi-infinite) matrix chain model:

$$\begin{aligned} Z_N &= \int \prod_{\alpha \le q} \mathrm{d}M_\alpha e^{-\,\mathrm{tr} \sum_{\alpha \le q} \left(V(M_\alpha) - M_{\alpha-1} M_\alpha\right)} \\ &= \int \prod_{\substack{\alpha \le q \\ i = 1 \ldots N}} \mathrm{d}\lambda_i^{(\alpha)} e^{-\sum_{i,\alpha} \left(V(\lambda_i^{(\alpha)}) - \lambda_i^{(\alpha)} \lambda_i^{(\alpha+1)}\right)} \Delta_N(\lambda^{(q)})\,. \end{aligned} \tag{4.1}$$

[4] This amplitude was considered independently in [3], where in addition its interpretation in the matrix model was understood.

[5] This observation leads us to the following point. As pointed out in [2], it is natural to associate a euclidean brane with boundary tachyon

$$T(\tilde{X}^0) = \hat{\lambda} \cos \tilde{X}^0$$

with the eigenvalue tunneling trajectory with turning point at $z = \hat{\lambda}$. Tensoring this boundary state in the spacelike $\tilde{X}^0$ theory with the D-particle state, one finds a wavefunction in the eigenvalue space which describes exactly this process.

where for the potential we take $V(M) = -\operatorname{Tr} M^2 + g \operatorname{Tr} M^4$ with g very small; its dependence will drop out in the double scaled theory. Upon taking the continuum limit and performing a Wick rotation, the above discretized expression reduces to a path-integral over the trajectories of the N eigenvalues

$$Z_N(\tau) = \int \prod_{i=1}^{N} D\lambda_i(t) e^{\frac{i}{\hbar}\int^\tau \mathrm{d}t \sum_{i=1}^{N}\left(\dot{\lambda}_i^2 - V(\lambda_i)\right)} \Delta_N(\tau)\,. \tag{4.2}$$

This partition function represents a quantum mechanical wave-function of the N eigenvalues $\lambda_i = \lambda_i(\tau)$ at the time-instant τ and satisfies the Schrodinger equation (here $p_i = -i\hbar \frac{\partial}{\partial \lambda_i(\tau)}$)

$$i\hbar \frac{d}{d\tau} Z_N(\tau) = \hat{H}_{old}(p,\lambda) Z_N(\tau) \tag{4.3}$$

with[6]

$$\hat{H}_{old}(p,\lambda) = \Delta_N^{-1} \hat{H}_N \Delta_N\,, \qquad \hat{H}_N = \sum_i \left(\frac{1}{4}p_i^2 + V(\lambda_i)\right). \tag{4.4}$$

We wish to find a new hamiltonian H_{new} that represents the time-evolution of the N eigenvalues in the presence of an extra probe eigenvalue. The way in which we will do this is motivated by the way one treats D-particles in string theory. Initially, the D-particle follows a classical trajectory,

$$\lambda_{N+1}(t) \equiv z(t)\,. \tag{4.5}$$

Eventually, to cancel string divergences, one is instructed to promote z to a quantum mechanical degree of freedom, and accordingly perform the path integral over all possible trajectories $z(t)$, weighted by an appropriate world-line action. After doing this path-integral, the extra D-particle has become fully quantized and indistinguishable from all other quantized D-particles. So our definition of the wave-function $Z_N(\lambda; z(\tau))$ of the N eigenvalues in the presence of the probe z must be such that:

$$Z_{N+1}(\tau) = \int Dz(t)\, e^{\frac{i}{\hbar}\int^\tau \mathrm{d}t\left(\dot{z}^2 - V(z) + \hbar \dot{z} A_B(z)\right)} Z_N(\lambda; z(t))\,. \tag{4.6}$$

Here we are including the possibility of a Berry phase term in the classical probe action, to be specified below. Its string theory interpretation is that the probe D-particle moves in a non-trivial closed string tachyon background, and this may induce such a term. Let us denote

$$e^{i\int^\tau \mathrm{d}t\, \dot{z} A_B(z)} = e^{i\varphi_B(z)}\,. \tag{4.7}$$

The Berry connection is defined such that

$$\left(\frac{\partial}{\partial z} + iA_B(z)\right) \tilde{Z}_N(\lambda; z) = 0 \tag{4.8}$$

[6]Note that the power of the Vandermonde in this equation is the inverse of what one would have naively expected by looking at (4.2). The correct kinetic term of the hamiltonian is obtained by restricting the U(N) invariant laplacian to the space eigenvalues. The difference arises due to the fact that the canonical momentum p_i and is not equal to the velocity $\dot{\lambda}_i$ but related via $\dot{\lambda}_i = \Delta^{-1} p_i \Delta$.

with $Z_N(\lambda, z) = e^{-i\varphi_B} \tilde{Z}_N(\lambda, z)$. The associated Berry phase is the non-integrable (i.e. path-dependent) phase factor. Together with the interaction hamiltonian $H_{\rm int}$, it will ensure that, upon quantization, the probe eigenvalue z satisfies the proper Fermi statistics with the other eigenvalues.

From its definition (4.6), and using that Z_{N+1} is as given in (4.2) with N replaced by $N+1$, we find that the new matrix wave-function $Z_N(\lambda, z)$ in the presence of the probe eigenvalue z is given by

$$Z_N(\lambda; z) = e^{-i\varphi_B(z)} \int \prod_{i=1}^{N} D\lambda_i(t) e^{\frac{i}{\hbar}\int^\tau {\rm d}t \sum_{i=1}^N \left(\dot{\lambda}_i^2 - V(\lambda_i)\right)} \Delta_{N+1}(\tau)\,. \tag{4.9}$$

This new wavefunction satisfies the Schrodinger equation

$$i\hbar \frac{d}{d\tau} Z_N(\lambda; z) = \hat{H}_{\rm new}(p, \lambda) Z_N(\lambda; z) \tag{4.10}$$

with

$$\hat{H}_{\rm new} = \Delta_{N+1}^{-1}\, \hat{H}_N\, \Delta_{N+1} \tag{4.11}$$

where we used the property (4.8) of the Berry phase. To extract the interaction hamiltonian, we should compare the new and old Schrodinger equations. We can write

$$\hat{H}_{\rm new} = \Delta_N^{-1}(\hat{H}_N + \hat{H}_{\rm int})\Delta_N \tag{4.12}$$

where (using that $\hbar$ is small) the interaction hamiltonian is given by[7]

$$\hat{H}_{\rm int}(z) = [\hat{H}_N, \hat{\Phi}(z(t))] \tag{4.13}$$

with

$$\hat{\Phi}(z) = \log\left(\frac{\Delta_{N+1}(z)}{\Delta_N}\right) = \sum_{i=1}^{N} \log(\lambda_i - z)\,. \tag{4.14}$$

This is the interaction hamiltonian used in [2]. The operator $\hat{\Phi}(z)$ is the Laplace transform of the macroscopic loop operator in the $c = 1$ matrix model. In expectation values, it creates a hole in the large-N diagrams, dual to the closed string worldsheets, with Dirichlet boundary conditions in both the time and the eigenvalue direction.

[7]We take this opportunity to note that the terms we have ignored in writing (4.13) are the matrix model description of the contact terms involved in exponentiating the boundary state into a shift in the closed-string background [26]. In this case, they have a very simple effect. The expression $\hat{H}_N + \hat{H}_{\rm int} = e^{\hat{\Phi}} \hat{H}_N e^{-\hat{\Phi}}$ says that

$$\hat{H}_{\rm int} = [\hat{\Phi}, \hat{H}_N] + \frac{1}{2}[\hat{\Phi}, [\hat{\Phi}, \hat{H}_N]] + \cdots\,.$$

But $[\hat{\Phi}, \hat{H}_N] = i\partial_t \hat{\Phi}$ is the field momentum of Φ, and hence the second term $[\hat{\Phi}, [\hat{\Phi}, \hat{H}_N]]$ is a c-number, and as a result all successive terms vanish. Thus, the effect is merely a multiplicative renormalization.

JHEP01(2004)039

4.2 Bouncing classical eigenvalue as a D-brane

In [2], the interaction hamiltonian $H_{\rm int}$ was used to compute the particle production due to an extra classical rolling eigenvalue $z(\tau) = \lambda e^{\tau}$. It was found that the result corresponds with the closed string production due to a rolling tachyon on the extended D-brane [14], provided one uses the adiabatic approximation discussed in section 3.3

This is a striking result for various reasons. First, while one could hope that the eigenvalues have a direct relation with the D-branes of the dual string theory, there exists no ironclad reasoning that would guarantee a simple, direct correspondence (see section 6). Secondly, one would normally expect [27, 28, 3] that the eigenvalues should correspond to the point- or particle-like D-brane states [17]. In the next section, when we turn to the quantum mechanical probe, we will reconcile our classical probe result with this viewpoint [3]. First, we will present further evidence supporting the identification of the classical probe trajectories with the D-string boundary state.

Consider the one-parameter family of classical bounce trajectories

$$z(t) = \bar{\lambda}\sqrt{\mu}\cosh t \tag{4.15}$$

labelled by the parameter $\bar{\lambda} \in [0, 1]$. The limit $\bar{\lambda} \downarrow 0$ (when defined appropriately[8]) describes the rolling trajectory $z(t) = \lambda e^t$ considered in [2]; the case $\bar{\lambda} = 1$ describes an eigenvalue that skims the Fermi sea.

In first order time-dependent perturbation theory, the emission amplitude for closed string excitations is given by

$$\mathcal{A}(\omega) = \int dt\, e^{i\omega t} \langle \mu_F + \omega | H_{\rm int} | \mu_F \rangle\,. \tag{4.16}$$

The relevant matrix element of the macroscopic loop operator takes the form [25]

$$\langle \mu_F + \omega | \hat{\Phi}(z(t)) | \mu_F \rangle = e^{i\delta(\omega)} \frac{\cos \pi\omega s(t)}{\omega \sinh \pi\omega} \tag{4.17}$$

where $s(t)$ parameterizes the probe trajectory via

$$z(t) = \sqrt{\mu}\cosh \pi s(t)\,. \tag{4.18}$$

The emission amplitude thus becomes

$$\mathcal{A}(\omega) = \int dt\, e^{i\delta(\omega) - i\omega t} \frac{\cos \pi\omega s(t)}{\sinh \pi\omega}\,. \tag{4.19}$$

This result must be compared with the CFT prescription as given in section 3.3 of the bouncing tachyon process on the D-string: the expression (4.19) indeed precisely matches with the CFT production amplitude (3.10).

An immediate puzzle raised by this correspondence, however, is that, as discussed in section 3.4, the D-string is expected to support a continuous spectrum of open string excitations. (See however the comment below eqn (3.16).) Can we find a signal betraying the

[8]Specifically, consider the trajectory $z_{t_0}(t) = \bar{\lambda}\sqrt{\mu}\cosh(t - t_0)$, and fix $z_{t_0}(0) = \bar{\lambda}\cosh t_0$ when taking $\bar{\lambda} \to 0,\ t_0 \to \infty$.

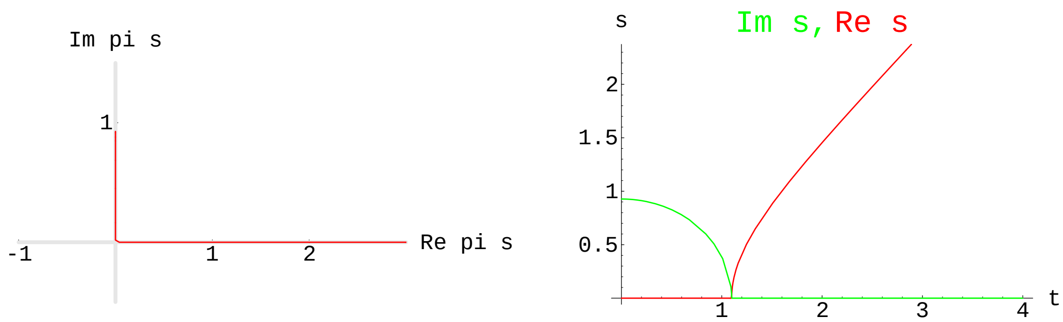

Figure 1: As t varies from zero to infinity, the uniformizing variable s takes a tour of the complex plane. Depicted is the trajectory $\cosh \pi s = 0.6 \cosh t$.

presence of these open strings directly from the matrix model amplitude (4.19)? In particular, since the closed string one-point amplitude is in fact an open string one-loop diagram, one may expect an imaginary contribution coming from the possible pair production of on-shell open strings.

Let us look more closely at the time-dependence of the uniformization parameter $s(t)$ that appears in (4.19). Combining (4.18) and (4.15), we have

$$\cosh \pi s(t) = \hat{\lambda} \cosh t \,. \tag{4.20}$$

Clearly, as long as $\hat{\lambda} \cosh t$ is larger than 1, the parameter $s(t)$ is real. But there is a short time interval (of order $\log \hat{\lambda}$ for small λ) for which $\hat{\lambda} \cosh t$ is less than 1, and $s(t)$ becomes imaginary, see figure 1.

We should note that the integrand in (4.19) is purely oscillatory, and the integral is therefore not unambiguously defined. (The same remark applies of course to formula (3.10).) Let us nevertheless proceed and change the variable of integration from t to s, with $t = t(s)$ defined by inverting (4.20). We can then write the amplitude (4.19) as an integral over a judiciously chosen contour in the complex s plane, as follows

$$\mathcal{A}(\omega) = \int ds\, \rho(s) e^{i\delta(\omega) - i\omega t(s)} \frac{\cos \pi\omega s}{\sinh \pi\omega} \,. \tag{4.21}$$

We now pick up the density

$$\rho(s) = \frac{dt}{ds} = \frac{\pi \sinh \pi s}{\sqrt{\cosh^2 \pi s - \hat{\lambda}^2}} \,. \tag{4.22}$$

Here $\rho(s)$ can be thought of as a probability density $|\psi(s)|^2$, since it is proportional to the amount of time the eigenvalue spends at the location s. The above expression for $\rho(s)$ is very similar to the naive formula (3.20) for the open string spectral density for the tachyon bounce. The two formulas differ, however, by an interchange of cosh and sinh-functions. Still, their formal similarity is a suggestive piece of evidence relating the matrix model to the continuum string theory, and may be of help in finding the proper dictionary between the two. It will be worthwhile to explore this particular correspondence further.

5. Rolling eigenvalues: quantum treatment

The most convenient representation of the $c = 1$ matrix model is as a free 2-d fermion field theory [29]. In this section, we will describe how, via standard path-integral quantization, our classical probe eigenvalue can indeed be turned into a free quantum mechanical fermionic particle.

Let us construct the free fermion field operator from the matrix model. It is defined as the second quantized operator that creates or destroys an eigenvalue: if we take its expectation value in the $N \times N$ matrix integral as given in (4.2), it turns it into an $N + 1 \times N + 1$ matrix integral (of the same form (4.2) but with Δ_N replaced by Δ_{N+1}), where it is understood that $\lambda_{N+1}(t) = z(t)$ starts (or ends) its life at $t = t_0$. Following the same steps as presented in the previous section, we can peel off the $N \times N$ expectation value from both sides, and obtain the following operator identity:

$$\Psi(z_0, t_0) = \int_{z(t_0)=z_0} Dz(t) e^{\frac{i}{\hbar}\int_{t_0} dt\, \left(\dot{z}^2 - V(z) + \hbar A_B(z)\dot{z} + \hat{H}_{\rm int}(z)\right)} \tag{5.1}$$

with A_B and $H_{\rm int}$ as defined in (4.8) and (4.13). Both sides of the above equation should be read as quantum mechanical operators that act on the $N \times N$ model defined by (4.2).

The equation (5.1) has a simple physical meaning: to describe the quantum mechanical propagation of a particle, one can either use the second quantized language and create the initial condition by means of a local quantum field creating the particle, or one can sum over all classical trajectories for the particle starting at a given point, weighted by the classical action. In our context, as we will see shortly, the equivalence between these two descriptions will imply a very interesting relationship between the two types of D-branes of 2-d string theory.

The formula (5.1) is clearly a bosonization/fermionization formula. Let us make this more explicit. First we note that in fact

$$A_B(z) = i\partial_z \hat{\Phi}(z) \tag{5.2}$$

which together with the form (4.13) of $H_{\rm int}$ allows us to rewrite (5.1) as

$$\Psi(z_0, t_0) = \int_{z(t_0)=z_0} Dz(t) e^{\frac{i}{\hbar}\int_{t_0} dt\, \left(\dot{z}^2 - V(z) + i\frac{d}{dt}\hat{\Phi}(z)\right)}\,, \tag{5.3}$$

which can be further simplified to

$$\Psi(z_0, t_0) = f(z_0, t_0)\exp\left(\frac{1}{\hbar}\hat{\Phi}(z_0, t_0)\right). \tag{5.4}$$

Here $f(z_0, t_0)$ is a wave-function satisfying the single particle Schrodinger equation with $H_0 = \frac{1}{4}p^2 + V(z)$, and the second factor is the standard vertex operator expression for a bosonized fermion.

At this point, let us recall our discussion of classical and quantum D-branes in section 1.1 of the Introduction. Comparing with the above results (5.1) and (5.4), we see that the operator $\Psi(z, t)$ in (5.4) acts like the second quantized field operator that creates the quantum version of the D-string. The form of the relation (5.4) further suggests that we should interpret the loop operator $\hat{\Phi}(z,t)$ as the local quantum field obtained by bosonization of the fermionic field $\Psi(z,t)$. There is a subtlety in this relationship, however. The standard collective field $\phi(z,t)$ is related to the eigenvalue density $\rho(z,t) = \mathrm{tr}\,\delta(z - Y(t))$ by $\rho(z,t) = \partial_z\phi(z,t)$. Therefore, comparing with the definition (4.14) of the loop operator $\hat{\Phi}$, we see that the collective field is obtained by taking the discontinuity of $\hat{\Phi}$ across the branch cut,

$$2\pi i\phi(z) = \hat{\Phi}(z + i\epsilon) - \hat{\Phi}(z - i\epsilon)\,, \tag{5.5}$$

turning the logarithm into a step function. By applying this formula to our first-quantized bosonization prescription, we can thus extract the local fermionic operator $\psi(z,t)$ which creates a quantum eigenvalue via

$$\psi(z,t) = e^{2\pi i\phi(z,t)}\,. \tag{5.6}$$

In this way, we recover the second quantized picture of the creation of an eigenvalue used in [3].

5.1 A quantum descent relation

The above results imply a remarkable relationship when translated back into the language of 2-d string theory.

Recall that excitations of the collective scalar field $\phi(z,t)$ correspond (up to a semi-local field redefinition due to the legpole factor) to excitations of the closed string tachyon. Following [3], we can go to the asymptotic region, where the eigenvalue motion becomes relativistic and only depends on one light-cone coordinate τ, and expand the field in terms of tachyon creation operators as

$$\phi(\tau) = \int \frac{dp}{\sqrt{2\pi|p|}} a_p^\dagger e^{-ip\tau}\,. \tag{5.7}$$

We can now make the comparison with the on-shell emission amplitude (3.23) due to a tachyon bounce on the localized D-brane described in section 3.5. This emission amplitude is given by

$$\phi(\tau) = \int \frac{dp}{2\pi\sqrt{2|p|}} a_p^\dagger \langle p|B_{ZZ}\rangle\langle\omega|B_\lambda\rangle^{HH}_{\mathrm{bounce}}\,, \tag{5.8}$$

which indeed exactly matches with the matrix model prediction, provided one identifies $\tau = \log\hat{\lambda}$. This identification was first obtained in [3].

When combined with the relationship (5.5), this identification leads to another interesting conclusion. As we saw before, the matrix element (4.17) of the loop operator coincides with the boundary state wavefunction of the D-string. The D-string wavefunction indeed reveals the same analytic structure when treated as a function of $z = \mu_B = \sqrt{\mu}\cosh\pi s$: as pointed out in [9, 30] it is multivalued on the μ_B plane, and the discontinuity is equal to the wavefunction of the D-particle

$$|B_{ZZ}\rangle = |B_{s=2i}\rangle - |B_{s=0}\rangle\,. \tag{5.9}$$

This equation should be compared with (5.5), and can be viewed as a descent relation that views the $s = 2i$ D-string as a bound state of the $s = 0$ D-string with the D-particle. Thus things fit together nicely: the procedure that extracts from the loop operator, that encodes the presence of a classical probe eigenvalue, the local quantum field $\psi(z,t)$, that creates and destroys eigenvalues, matches precisely with the descent relation between the D-string and the D-particle [30]. This further supports the interpretation, put forward in [3], of $\psi(z,t)$ as the field that creates the ZZ D-particle.

6. Conclusion

We would like to conclude this paper by addressing a few possibly confusing points. All three issues revolve around the fact that D-branes can dissolve into a closed string background.

6.1 The raising of the Fermi sea

The result section 5 seems to indicate that the fermion field operator $\Psi(z_0, t_0)$, that creates the eigenvalue, can be completely expanded in terms of perturbative closed string tachyon modes. This assertion can be true only in the strict $\hbar$ to 0 limit. For $\hbar$ finite, it is well-known that the fermion field and corresponding vertex operator $e^{i\Phi}$ are *not* part of the usual Fock space of the Φ-field. Φ is a periodic variable, and the total fermion number $Q_F = \int dz\, \partial_z \Phi$ defines a superselection sector: no finite number of bosonic oscillators can change Q_F. This statement is dual to fact that in 2-d string theory, no finite number of normalizable closed string tachyon modes can produce sufficient backreaction to induce a shift in the non-normalizable zero-mode (2.12).

6.2 Worldsheet renormalization

Based on experience and specific arguments [31, 32, 33] one expects that the study of off-shell string theory requires a violation of worldsheet conformal invariance. Therefore, the reader may be given pause by the adiabatic prescription of section 1.1, which apparently provides a prescription for studying arbitrary classical trajectories of D-particles *using worldsheet conformal field theory.*

The procedure outlined in section 1.1 was inspired by the $c = 1$ matrix model. Indeed, when applied to two-dimensional string theory it has produced results which we know independently to be correct.

The resolution of this puzzle lies in the fact that it is not those worldsheets whose conformal invariance is violated — this violation occurs on the worldsheets made from the large-N diagrams of the open string theory. These worldsheets clearly do not exhibit conformal invariance before the double-scaling limit. Taking the continuum limit implements the approach to the IR fixed point. One should be concerned, however, about the application to greater-than-two-dimensional models where it will be difficult to realize the higher-dimensional target space as a continuum limit of the lattice theory. In conclusion, the prescription we have advocated *does* involve non-conformal worldsheets, but it is via the large-N RG that one restores criticality.

6.3 The identity of an eigenvalue

By now, a number of quantitative matches have been made between calculations involving branes in 2-d string theory and calculations in the $c = 1$ matrix model [2, 3, 9]. Inevitably, one encounters the question: "which brane *is* the eigenvalue?"

In fact, this is not a very well-posed question. Indeed, while both types of D-branes appear to be very closely related to the eigenvalues, neither is a perfect match: the classical extended branes do seem to correspond to the classical eigenvalues, but have many more degrees of freedom, while the localized branes correspond to the quantum mechanical rather than classical eigenvalues. There is also no reason to expect a perfect match: the $N + 1$-st eigenvalue will interact differently with the background closed string theory defined by the collective field theory of the first N eigenvalues, than with the closed string background provided by, say, the first M eigenvalues. Following the philosophy of section 6 of [2], one could try to introduce the notion of the "bare" string theory, which is the underlying theory relative to which we define a decoupling limit of the unstable D-particles, by sending $g_s \to 0$ while populating it with $N \to \infty$ D-branes, keeping μ fixed. Since the Liouville direction is *generated* by the matrices, the "bare" string theory knows nothing of it.

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